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BMAP/BMSP/1 Queue with Randomly Varying Environment

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Abstract: This paper studies two stochastic batch Markovian arrival and batch Markovian service single server queue BMAP/BMSP/1 queue Models (A) and (B) with randomly varying k* distinct environments. The arrival process of the queue has matrix representation $\{D_m^i : 0 \le m \le M\}$ of order k_i describing the BMAP and the service process has matrix representation $\{S_n^i : 0 \le n \le N\}$ of order k'_i describing the BMSP respectively in the environment i for $1 \le i \le k^*$. Whenever the environment changes from i to j the arrival BMAP and service BMSP change from the i version to the j version with the exception of the first remaining arrival time and first remaining service time in the new environment start as per stationary probability vector of j version of the BMAP and of j version of the BMSP respectively for $1 \le i, j \le k^*$. The queue system has infinite storing capacity and the state space is identified as five dimensional one to apply Neuts' matrix methods. In the environment i, the sizes of the arrivals and services are governed by the matrices D_m^i and S_n^i, with respect to environment. The service process is stopped when the queue becomes empty and is started with initial probability vector of corresponding environment BMSP when the arrival occurs. Matrix partitioning method is used to study the models. In Model (A) the maximum of the arrival sizes is greater than the maximum of the service sizes and the infinitesimal generator is partitioned mostly as blocks of the sum of the products of BMAP arrival and BMSP service phases in the various environments times the maximum of the arrival sizes for analysis. In Model (B) the maximum of the arrival sizes is less than the maximum of the service sizes. The generator is partitioned mostly using blocks of the same sum-product of batch Markovian phases times the maximum of the service sizes. Block circulant matrix structure is noticed in the basic system generators. The stationary queue length probabilities, its expected values, its variances and probabilities of empty levels are derived for the two models using matrix methods. Numerical examples are presented for illustration.

Keywords: *Batch Arrivals, Batch Services, Block Circulant Matrix, Neuts Matrix Methods, Phase Type Distribution.*

I. INTRODUCTION

In this paper two batch arrival and batch service BMAP/BMSP/1 queues with random environment have been studied using matrix geometric methods. Numerical studies on matrix methods are presented by Bini, Latouche and Meini [1]. Multi server model has been of interest in Chakravarthy and Neuts [2]. Birth and death model has been analyzed by Gaver, Jacobs and Latouche [3]. Analytic methods are focused in Latouche and Ramaswami [4] and for matrix geometric methods one may refer Neuts [5]. For M/M/1 bulk queues with random environment models one may refer Rama Ganesan, Ramshankar and Ramanarayanan [6] and M/M/C bulk queues with random environment models are of interest in Sandhya, Sundar, Rama, Ramshankar and Ramanarayanan [7]. PH/PH/1 bulk queues without variation of environments have been treated by Ramshankar, Rama Ganesan and Ramanarayanan [8] and with variations of environment have been analyzed by Ramshankar,Rama, Sandhya, Sundar and Ramanarayanan [9]. BMAP/M/C queue with bulk service and random environment has been studied by Rama, Ramshankar, Sandhya, Sundar and Ramanarayanan [10]. The models considered here are general compared to existing models. Batch Markovian service process (BMSP) considered in this paper is similar to batch Markovian arrival process (BMAP) and BMAP has been studied by Lucantony[11] and has been analyzed further by Cordeiro and Kharoufch [12] . When the queue becomes empty the service process is stopped and it is started with starting probability vector when the arrivals occur in the empty queue. Usually bulk arrival models have M/G/1 upper-Heisenberg block matrix structure. The decomposition of a Toeplitz sub matrix of the infinitesimal generator is required to find the stationary probability vector and matrix geometric structures are rarely noted. In such analysis the recurrence relation method to find the stationary probabilities is stopped at a certain level in most general cases using a terminating method very well explained by Qi-Ming He [13] and this stopping limitation of terminating method converts an infinite arrival system to a finite arrival one. In special cases generating function has been identified by Rama and Ramanarayanan [14]. However the method of partitioning of the infinitesimal generator along with environment, BMAP phases and BMSP phases used in this paper is presenting matrix geometric solution for finite sized batch arrivals and batch services models. The M/PH/1 and PH/M/C queues with random environments have been studied by Usha [15] and [16] without bulk arrivals and bulk services. It has been noticed by Usha [15, 16] that when the environment changes the remaining arrival and service times are to be completed in the new environment. The residual arrival time and the residual service time distributions in the new environment are to be considered in the new environment at an arbitrary epoch since the spent arrival time and the spent service time have been in the previous environment with distinct sizes of PH phase. Further new arrival time and new service time from the start using initial PH distributions of the new environment cannot be considered since the arrival and the service have been partly completed in the previous environment indicating the stationary versions of the arrival and service distributions in the new environments are to be used for the completions of the residual arrival and service times in the new environment and on completion of the same the next arrival and service onwards they have initial versions of the PH distributions of the new environment. The stationary version of the distribution for residual time has been well explained in Qi-Ming He [13] where it is named as equilibrium PH distribution. Randomly varying environment BMAP/BMSP/1 queue models have not been treated so far at any depth.

Two models (A) and (B) on BMAP/BMSP/1 bulk queue systems with infinite storage space for customers are studied here using the block partitioning method. Model (A) presents the case when M, the maximum of the arrival sizes of BMAP is bigger than N, the maximum of the service sizes of BMSP. In Model (B), its dual case N is bigger than M, is treated. In general in Queue models, the state space of the system has the first co-ordinate indicating the number of customers in the system but here the customers in the system are grouped and considered as members of blocks of sizes of the maximum for finding the rate matrix. Using the maximum of the batch arrival sizes and of the batch service sizes and grouping the customers as members of blocks in addition to coordinates of the arrival and service phases for partitioning the infinitesimal generator is a new approach in this area. The matrices appearing as the basic system generators in these two models due to block partitioned structure are seen as block circulant matrices. The paper is organized in the following manner. In sections II and III the stationary probability of the number of customers waiting for service, the expectation and the variance and the probability of empty queue are derived for these Models (A) and (B). In section IV numerical cases are presented to illustrate them.

II. MODEL (A): MAXIMUM ARRIVAL SIZE M > MAXIMUM SERVICE SIZE N

2.1Assumptions

(i)There are k* environments. The environment changes as per changes in a continuous time Markov chain with infinitesimal generator Q_1 of order k* with stationary probability vector π'.

(ii)In the environment i for $1 \leq i \leq k^*$, the batch arrivals occur in accordance with Batch Markovian Arrival Process with matrix representation for the rates of batch sizes m given by the finite sequence $\{D_m^i, 0 \leq m \leq M_i\}$ with phase order k_i where D_0^i has negative diagonal elements and its other elements are non-negative; D_m^i has non-negative elements for $1 \leq m \leq M_i$ where M_i is the maximum batch arrival size in the environment i. Let $D^i = \sum_{m=0}^{M_i} D_m^i$ and φ_i be the stationary probability vector of the generator matrix D^i with $\varphi_i D^i = 0$ and φ_ie = 1.

(iii) In the environment i for $1 \leq i \leq k^*$, when the queue length L is more than or equal to the maximum batch service size N_i, $(L \geq N_i)$ of the environment, the batch services occur in accordance with Batch Markovian Service Process with matrix representation for the rates of batch sizes n given by the finite sequence $\{S_n^i, 0 \leq n \leq N_i\}$ with phase order k'_i where S_0^i has negative diagonal elements and its other elements are non-negative; S_n^i has non-negative elements for $1 \leq n \leq N_i$. Let $S^i = \sum_{n=0}^{N_i} S_n^i$ and Φ_i be the stationary probability vector of the generator matrix S^i with $\Phi_i S^i = 0$ and Φ_ie = 1.

(iv)When n customers n < N_i are waiting for service, then n' customers are served with rate S_n^i for $1 \leq n' \leq n-1$ and n customers are served with rate $\sum_{j=n}^{N_i} S_n^i = S'_{i,n}$ which is a matrix of order k'_i.

(v)The BMSP service process is stopped when the queue becomes empty and is started in the environment i for $1 \leq i \leq k^*$with initial probability vector $\beta_i$of the i version BMSP when the arrival occurs.

(v) When the environment changes from i to j for $1 \leq i, j \leq k^*$, the arrival process and service process in the new environment j start as per stationary (equilibrium) probability vectors φ_j of D^j and ϕ_j of S^j respectively of the j versions of arrival process BMAP, namely, { D_m^j, 0 ≤ m ≤ M_j } and of the j version of the service process, namely, BMSP {S_n^i, 0 ≤ n ≤ N_i} in the new environment.

(vii) The maximum batch arrival size of all BMAPs', M= $\max_{1\le i\le k*} M_i$ is greater than the maximum batch service size N= $\max_{1\le i\le k*} N_i$.

2.2.Analysis

The state of the system of the continuous time Markov chain X (t) under consideration is presented as follows. X(t) = {(0, i, j) : for 1 ≤ i ≤ k* and 1 ≤ j ≤ k_i)} U {(0, k, i, j, j') ; for 1 ≤ k ≤ M-1; 1 ≤ i ≤ k*; 1 ≤ j ≤ k_i; 1 ≤ j ≤ k'_i} U {(n, k, i, j, j'): for 0 ≤ k ≤ M-1; 1 ≤ i ≤ k*; 1 ≤ j ≤ k_i; 1 ≤ j ≤ k'_i and n ≥ 1}. (1)
The chain is in the state (0, i, j) when the number of customers in the queue is 0, the environment state is i for 1 ≤ i ≤ k* and the arrival BMAP phase is j for 1 ≤ j ≤ k_i. The chain is in the state (0, k, i, j, j') when the number of customers is k for 1 ≤ k ≤ M-1, the environment state is i for 1 ≤ i ≤ k*, the arrival BMAP phase is j for 1 ≤ j ≤ k_i and the service BMSP phase is j' for 1 ≤ j' ≤ k'_i. The chain is in the state (n, k, i, j, j') when the number of customers in the queue is n M + k, for 0 ≤ k ≤ M-1 and 1 ≤ n < ∞, the environment state is i for 1 ≤ i ≤ k*, the arrival BMAP phase is j for 1 ≤ j ≤ k_i and the service BMSP phase is j' for 1 ≤ j' ≤ k'_i. When the number of customers waiting in the system is r, then r is identified with (n, k) where r on division by M gives n as the quotient and k as the remainder. The chain X (t) describing model has the infinitesimal generator Q_A of infinite order which can be presented in block partitioned form given below.

$$Q_A=\begin{bmatrix} B_1 & B_0 & 0 & 0 & . & . & . & \cdots \\ B_2 & A_1 & A_0 & 0 & . & . & . & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & . & . & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & . & \cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad (2)$$

In (2) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}$, ... For partition purpose the zero states in the first two sets given in (1) are combined. The vector $\underline{0}$ is of type 1 x $[\sum_{i=1}^{k*} k_i + (M-1)\sum_{i=1}^{k*} k_i\, k'_i]$ and $\underline{0}$=((0,1,1),(0,1,2),(0,1,3)…(0,1, k_1),(0,2,1),(0,2,2),(0,2,3)…(0,2, k_2),……(0,k*,1),(0,k*,2),(0,k*,3)…(0,k*,k_{k*}), (0,1,1,1,1),(0,1,1,1,2)…(0,1,1,1, k'_1),(0,1,1,2,1),(0,1,1,2,2)…(0,1,1,2,k'_1),(0,1,1,3,1)...(0,1,1,3,k'_1)…(0,1,1,k_1,1) …(0,1,1,k_1, k'_1),(0,1,2,1,1),(0,1,2,1,2)…(0,1,2,1, k'_2),(0,1,2,2,1),(0,1,2,2,2)…(0,1,2,2,k'_2),(0,1,2,3,1)...(0,1,2,3, k'_2)…(0,1,2,k_2,1)…(0,1,2,k_2, k'_2),(0,1,3,1,1)…(0,1,3,k_3, k'_3)…(0,1,k*,1,1),…,(0,1,k*,k_{k*}, k'_{k*}),(0,2,1,1,1),(0,2 ,1,1,2)…(0,2,k*,k_{k*},k'_{k*}),(0,3,1,1,1)…(0,3,k*,$k_{k*}k_{k*}$),(0,4,1,1,1)…(0,4,k*,$k_{k*}k'_{k*}$)…(0,M-1,1,1,1)… (0,M-1,k*,k_{k*}, k'_{k*})). . (3)
The vector $\underline{n}$ is of type 1x$[M\sum_{i=1}^{k*} k_i\, k'_i]$ and is given in a similar manner as follows $\underline{n}$=(n,0,1,1,1),(n,0,1,1,2)…(n,0,1,1, k'_1),(n,0,1,2,1)…(n,0,1,2,k'_1)…(n,0,k*,1,1)…(n,0,k*,k_{k*}, k'_{k*}),(n,1,1,1,1)… (n,1,k*,k_{k*},k'_{k*}),(n,2,1,1,1)...(n,2,k*,k_{k*},k'_{k*})…(n,M-1,1,1,1),(n,M-1,1,1,2)…(n,M-1,k*,k_{k*}, k'_{k*})). (4)
The matrices B_1 and A_1 have negative diagonal elements, they are of orders $[\sum_{i=1}^{k*} k_i + (M-1)\sum_{i=1}^{k*} k_i\, k'_i]$ and $[M\sum_{i=1}^{k*} k_i\, k'_i]$ respectively and their off diagonal are non-negative. The matrices A_0 and A_2 have nonnegative elements and are of order $[M\sum_{i=1}^{k*} k_i\, k'_i]$. The matrices B_0 and B_2 have non-negative elements and are of types $[\sum_{i=1}^{k*} k_i + (M-1)\sum_{i=1}^{k*} k_i\, k'_i]$ x $[M\sum_{i=1}^{k*} k_i\, k'_i]$ and $[M\sum_{i=1}^{k*} k_i\, k'_i]$ x $[\sum_{i=1}^{k*} k_i + (M-1)\sum_{i=1}^{k*} k_i\, k'_i]$. Component matrices of A_i and B_i for i=0, 1, 2 are defined below. Let ⊕ and ⊗ denote the Kronecker sum and Kronecker product.
Let $Q'_i = D_0^i \oplus S_0^i + (Q_1)_{i,i} I_{k_i k'_i} = (D_0^i \otimes I_{k'_i}) + (I_{k_i} \otimes S_0^i) + (Q_1)_{i,i} I_{k_i k'_i}$ for 1 ≤ i ≤ k* (5)
where I indicates the identity matrices of orders given in the suffixes, Q'_i is of order $k_i k'_i$ and the last term is a diagonal matrix of order $k_i k'_i$. Considering the stationary probability starting vectors of BMAP and of BMSP for the change of environment the following matrix Ω of order $\sum_{i=1}^{k*} k_i\, k'_i$ is defined which is concerned with change of environment during arrival and service time.

$$\Omega=\begin{bmatrix} Q'_1 & \Omega_{1,2} & \Omega_{1,3} & \cdots & \Omega_{1,k*} \\ \Omega_{2,1} & Q'_2 & \Omega_{2,3} & \cdots & \Omega_{2,k*} \\ \Omega_{3,1} & \Omega_{3,2} & Q'_3 & \cdots & \Omega_{3,k*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega_{k*,1} & \Omega_{k*,2} & \Omega_{k*,3} & \cdots & Q'_{k*} \end{bmatrix} \quad (6)$$

where $\Omega_{i,j}$ is a rectangular matrix of type $k_i k'_i$ x $k_j k'_j$ and all its rows are equal to $(Q_1)_{i,j}$ $(\varphi_j \otimes \phi_j)$ for i ≠ j , 1 ≤ i, j ≤ k*.The matrix of arrival rates of n customers corresponding to the arrival in BMAP in the environment i is D_n^i which is a matrix of order k_i with non-negative elements for 1 ≤ n ≤ M_i and D_n^i = 0 matrix for n > M_i for 1 ≤ i ≤ k*. (7)

The matrix of service rates of n customers corresponding to the service in BMSP in the environment i is S_n^i which is of order $k'_i$with non-negative elements for $1 \le n \le N_i$ and S_n^i = 0 matrix for n > N_i for $1 \le i \le k^*$. (8)

$$\text{Let } \Lambda_n = \begin{bmatrix} D_n^1 \otimes I_{k'_1} & 0 & 0 & \cdots & 0 \\ 0 & D_n^2 \otimes I_{k'_2} & 0 & \cdots & 0 \\ 0 & 0 & D_n^3 \otimes I_{k'_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_n^{k*} \otimes I_{k'_{k*}} \end{bmatrix} \text{ for } 1 \le n \le M \qquad (9)$$

In (9) Λ_n is a square matrix of order $\sum_{i=1}^{k*} k_i\, k'_i$; $D_n^j \otimes I_{k'_j}$ is a square matrix of order $k_j k'_j$ for $1 \le j \le k^*$ and D_n^j =0 matrix for n > M_j.The (i, j) component 0 appearing in (9) is a block zero matrix of type $k_i k'_i$ x $k_j k'_j$.

$$\text{Let } U_n = \begin{bmatrix} I_{k_1} \otimes S_n^1 & 0 & 0 & \cdots & 0 \\ 0 & I_{k_2} \otimes S_n^2 & 0 & \cdots & 0 \\ 0 & 0 & I_{k_3} \otimes S_n^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{k_{k*}} \otimes S_n^{k*} \end{bmatrix} \text{ for } 1 \le n \le N \qquad (10)$$

In (10) U_n is a square matrix of order $\sum_{i=1}^{k*} k_i\, k'_i$; $I_{k_j} \otimes S_n^j$ is a square matrix of order $k_j k'_j$ for $1 \le j \le k^*$ and 0 appearing as (i, j) component of (10) is a block zero rectangular matrix of type $k_i k'_i$ x $k_j k'_j$ .The matrix A_i for i = 0,1,2 are as follows.

$$A_0 = \begin{bmatrix} \Lambda_M & 0 & \cdots & 0 & 0 & 0 \\ \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 & 0 \\ \Lambda_{M-2} & \Lambda_{M-1} & \cdots & 0 & 0 & 0 \\ \Lambda_{M-3} & \Lambda_{M-2} & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \Lambda_3 & \Lambda_4 & \cdots & \Lambda_M & 0 & 0 \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M \end{bmatrix} \qquad (11)$$

$$A_2 = \begin{bmatrix} 0 & \cdots & 0 & U_N & U_{N-1} & \cdots & U_2 & U_1 \\ 0 & \cdots & 0 & 0 & U_N & \cdots & U_3 & U_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & U_N & U_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U_N \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad (12)$$

$$A_1 = \begin{bmatrix} \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \Lambda_{M-N} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ U_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ U_2 & U_1 & \Omega & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_N & U_{N-1} & U_{N-2} & \cdots & \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ 0 & U_N & U_{N-1} & \cdots & U_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ 0 & 0 & U_N & \cdots & U_2 & U_1 & \Omega & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_N & U_{N-1} & U_{N-2} & \cdots & \Omega & \Lambda_1 \\ 0 & 0 & 0 & \cdots & 0 & U_N & U_{N-1} & \cdots & U_1 & \Omega \end{bmatrix} \qquad (13)$$

For defining the matrices B_i for i = 0,1,2 the following component matrices are required

$$\Lambda'_n = \begin{bmatrix} D_n^1 \otimes \beta_1 & 0 & 0 & \cdots & 0 \\ 0 & D_n^2 \otimes \beta_2 & 0 & \cdots & 0 \\ 0 & 0 & D_n^3 \otimes \beta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_n^{k*} \otimes \beta_{k*} \end{bmatrix} \text{ for } 1 \le n \le M \qquad (14)$$

Λ'_n is a rectangular matrix of type ($\sum_{i=1}^{k*} k_i$)x $\sum_{i=1}^{k*}(k_i k'_i)$ for $1 \le n \le M$; $D_n^i \otimes \beta_i$ is a rectangular matrix of order $k_i x k_i k'_i$ and 0 appearing as (i, j) component of (14) is a block zero rectangular matrix of type k_i x $k_j k'_j$ for $1 \le i, j \le k^*$. Let $V'_{i,n} = I_{k_i} \otimes S'_{i,n+1}e$ for $1 \le n \le N-1$ is a matrix of type $k_i k'_i$ x k_i for $1 \le i \le k^*$ where $S'_{i,n} = \sum_{j=n}^{N} S_j^i$ for $1 \le j \le N$ and let

$$V_n = \begin{bmatrix} V'_{1,n} & 0 & 0 & \cdots & 0 \\ 0 & V'_{2,n} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & V'_{k*,n} \end{bmatrix} \text{ for } 1 \le n \le N-1. \qquad (15)$$

This is a rectangular matrix of type ($\sum_{i=1}^{i=k*} k_i k'_i) \, x \, \left(\sum_{i=1}^{k*} k_i\right)$ and 0 appearing in the (i, j) component is a rectangular 0 matrix of type $k_i k'_i$ x k_j for $1 \le i, j \le k^*$.

Let U = $\begin{bmatrix} I_{k_1} \otimes S'_{1,1}\mathrm{e} & 0 & 0 & \cdots & 0 \\ 0 & I_{k_2} \otimes S'_{2,1}\mathrm{e} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{k_1} \otimes S'_{k*,1}\mathrm{e} \end{bmatrix}$ (16)

In (16), U is a rectangular matrix of type ($\sum_{i=1}^{i=k*} k_i k'_i$) x $\left(\sum_{i=1}^{k*} k_i\right)$ and 0 appearing in the (i, j) component is a rectangular 0 matrix of type $k_i k'_i$ x k_j for 1 ≤ i, j ≤ k*. $I_{k_i} \otimes S'_{i,n}$ is a rectangular matrix of type $k_i k'_i\ x\ k_i$ for 1 ≤ i ≤ k*.The matrix B_0 is same as that of A_0 when Λ_M in the first row of A_0 is replaced byΛ'_M. The matrix B_1 is given below. The matrix B_2 is same as that of $A_2$when the first block column with 0 is considered as $\sum_{i=1}^{k*} k_i$ columns block instead of $\sum_{i=1}^{k*} k_i k'_i$columns block of A_2. To write $B_1$the block for $\underline{0}$ is to be considered which has queue length, L= 0, 1, 2…M-1. When L = 0 there is only arrival process and no service process. The change in environment from i to j switches BMAP j version as started by its stationary probability vector in the new environment for 1 ≤ i ≠ j, ≤ k*. When an arrival occurs and queue length becomes L in the environment i next arrival time starts and the service time starts with starting probability vector β_i for 1 ≤ i ≤ k*. In the $\underline{0}$ when L =1, 2, …, M-1 all the processes arrival, service and environment are active as in other blocks $\underline{n}$ for n > 0. Considering the change of environment switches on the BMAP as per stationary probability vector in the new environment when the queue is empty, the following matrix Ω' of order $\sum_{i=1}^{k*} k_i$is defined which is concerned with change of environment during arrival.

Ω'= $\begin{bmatrix} T'_1 & \Omega'_{1,2} & \Omega'_{1,3} & \cdots & \Omega'_{1,k*} \\ \Omega'_{2,1} & T'_2 & \Omega'_{2,3} & \cdots & \Omega'_{2,k*} \\ \Omega'_{3,1} & \Omega'_{3,2} & T'_3 & \cdots & \Omega'_{3,k*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega'_{k*,1} & \Omega'_{k*,2} & \Omega'_{k*,3} & \cdots & T'_{k*} \end{bmatrix}$ (17)

Here $T'_i = D_0^i + (Q_1)_{i,i} I_{k_i}$ and $\Omega'_{i,j}$ is a rectangular matrix of type k_i x $k_j$whose all rows are equal to $(Q_1)_{i,j}\ \varphi_j$ presenting the rates of changing to phases in the new environment for i ≠ j and 1 ≤ i, j ≤ k*.

$$B_1 = \begin{bmatrix} \Omega' & \Lambda'_1 & \Lambda'_2 & \cdots & \Lambda'_{M-N-2} & \Lambda'_{M-N-1} & \Lambda'_{M-N} & \cdots & \Lambda'_{M-2} & \Lambda'_{M-1} \\ U & \Omega & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ V_1 & U_1 & \Omega & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ V_{N-1} & U_{N-1} & U_{N-2} & \cdots & \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ 0 & U_N & U_{N-1} & \cdots & U_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ 0 & 0 & U_N & \cdots & U_2 & U_1 & \Omega & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_N & U_{N-1} & U_{N-2} & \cdots & \Omega & \Lambda_1 \\ 0 & 0 & 0 & \cdots & 0 & U_N & U_{N-1} & \cdots & U_1 & \Omega \end{bmatrix} \quad (18)$$

Q'_A =

$$\begin{bmatrix} \Omega+\Lambda_M & \Lambda_1 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \Lambda_{M-N}+U_N & \cdots & \Lambda_{M-2}+U_2 & \Lambda_{M-1}+U_1 \\ \Lambda_{M-1}+U_1 & \Omega+\Lambda_M & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3}+U_3 & \Lambda_{M-2}+U_2 \\ \Lambda_{M-2}+U_2 & \Lambda_{M-1}+U_1 & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4}+U_3 & \Lambda_{M-3}+U_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_{M-N+2}+U_{N-2} & . & \cdots & . & . & . & \cdots & \Lambda_{M-N}+U_N & \Lambda_{M-N+1}+U_{N-1} \\ \Lambda_{M-N+1}+U_{N-1} & . & \cdots & . & . & . & . & \Lambda_{M-N-1} & \Lambda_{M-N}+U_N \\ \Lambda_{M-N}+U_N & . & \cdots & \Omega+\Lambda_M & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ \Lambda_{M-N-1} & \Lambda_{M-N}+U_N & \cdots & \Lambda_{M-1}+U_1 & \Omega+\Lambda_M & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-2}+U_2 & \Lambda_{M-1}+U_1 & \Omega+\Lambda_M & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_{M-N}+U_N & \Lambda_{M-N+1}+U_{N-1} & \Lambda_{M-N+2}+U_{N-2} & \cdots & \Omega+\Lambda_M & \Lambda_1 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-1} & \Lambda_{M-N}+U_N & \Lambda_{M-N+1}+U_{N-1} & \cdots & \Lambda_{M-1}+U_1 & \Omega+\Lambda_M \end{bmatrix} \quad (19)$$

The basic generator of the bulk queue which is concerned with only the arrival and service is a matrix of order [$M \sum_{i=1}^{k*} k_i\, k'_i$] given above in (19) where $Q'_A = A_0 + A_1 + A_2$ (20)

Its probability vector w gives, wQ'_A =0 and w. e = 1 (21)

It is well known that a square matrix in which each row (after the first) has the elements of the previous row shifted cyclically one place right, is called a circulant matrix. It is very interesting to note that the matrix Q'_A is a block circulant matrix where each block matrix is rotated one block to the right relative to the preceding block partition. In (19), the first block-row of type [$\sum_{i=1}^{k*} k_i\, k'_i$] x[$M \sum_{i=1}^{k*} k_i\, k'_i$] is, $W = (\Omega + \Lambda_M, \Lambda_1, \Lambda_2,$ $\ldots, \Lambda_{M-N-2}, \Lambda_{M-N-1}, \Lambda_{M-N} + U_N, \ldots, \Lambda_{M-2} + U_2, \Lambda_{M-1} + U_1)$ which gives as the sum of the blocks ($\Omega +$ *ΛM+ Λ1+ Λ2 +…+ΛM−N−2+ ΛM−N−1+ΛM−N+UN+…+ΛM−2+U2+ ΛM−1+U1*= Ω'' which is the matrix given by

$$\Omega''=\begin{bmatrix} Q''_1 & \Omega_{1,2} & \Omega_{1,3} & \cdots & \Omega_{1,k*} \\ \Omega_{2,1} & Q''_2 & \Omega_{2,3} & \cdots & \Omega_{2,k*} \\ \Omega_{3,1} & \Omega_{3,2} & Q''_3 & \cdots & \Omega_{3,k*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Omega_{k*,1} & \Omega_{k*,2} & \Omega_{k*,3} & \cdots & Q''_{k*} \end{bmatrix} \quad (22)$$

where using (5) and (6), $Q''_i = (D^i \otimes I_{k'_i}) + (I_{k_i} \otimes S^i) + (Q_1)_{i,i} I_{k_i k'_i}$ for $1 \le i \le k*$. The stationary probability vector of the basic generator given in (19) is required to get the stability condition. Consider the vector w = ($\pi'_1\varphi_1 \otimes \phi_1$, $\pi'_2\varphi_2 \otimes \phi_2$,…, $\pi'_{k*}\varphi_{k*} \otimes \phi_{k*}$) where π' = $(\pi'_1, \pi'_2, \ldots, \pi'_{k*})$ is the stationary probability vector of the environment, φ_i *and* ϕ_i are the stationary probability vectors of the i version BMAP and i version BMSP D^i and S^irespectively. It may be noted $\pi'_i(\varphi_i \otimes \phi_i)[(D^i \otimes I_{k'_i}) + (I_{k_i} \otimes S^i)]$ =0. This gives $\pi'_i(\varphi_i \otimes \phi_i)Q''_i = \pi'_i(Q_1)_{i,i}\ (\varphi_i \otimes \phi_i)\ I\ = \pi'_i(Q_1)_{i,i}\ (\varphi_i \otimes \phi_i)$ for $1 \le i \le k*$. Now the first column of the matrix multiplication of wΩ'' is $\pi'_1(Q_1)_{1,1}\varphi_{1,1}\phi_{1,1} + \pi'_2 (Q_1)_{2,1}\varphi_{11}\phi_{11}[(\varphi_2 \otimes \phi_2)e]$ +.....+ $\pi'_{k*} (Q_1)_{k*,1}\varphi_{11}\phi_{11}[(\varphi_{k*} \otimes \phi_{k*})]e = 0$ since $(\varphi_i \otimes \phi_i)e = 1$ and $\pi' Q_1$=0. In a similar manner it can be seen that the first column block of wΩ'' is $\pi'_1(Q_1)_{1,1}\varphi_1 \otimes \phi_1 + \pi'_2 (Q_1)_{2,1}\varphi_1 \otimes \phi_1[(\varphi_2 \otimes \phi_2)e]$ +.....+ $\pi'_{k*} (Q_1)_{k*,1}\varphi_1 \otimes \phi_1[(\varphi_{k*} \otimes \phi_{k*})]e = 0$ and i-th column block is $\pi'_1(Q_1)_{1,i}\varphi_i \otimes \phi_i[(\varphi_1 \otimes \phi_1)e] + \pi'_2 (Q_1)_{2,i}\varphi_i \otimes \phi_i[(\varphi_2 \otimes \phi_2)e]$ +.....+ $\pi'_i (Q_1)_{i,i}\varphi_i \otimes \phi_i + \ldots + \pi'_{k*} (Q_1)_{k*,i}\varphi_i \otimes \phi_i[(\varphi_{k*} \otimes \phi_{k*})]e$= 0. This shows that $w(\Omega + \Lambda_M) + w\Lambda_1 + w\Lambda_2 + \ldots + w\Lambda_{M-N-2} + w\Lambda_{M-N-1} + w\Lambda_{M-N} + wU_N + \ldots + w\Lambda_{M-2} + wU_2 + w\Lambda_{M-1} + wU_1$= w Ω''=0. So (w, w,…,w) .W= 0 = (w, w, ….w) W' where W' is the transpose W. This shows (w,w...w) is the left eigen vector of Q'_A and the corresponding probability vector is w' = $\left(\frac{w}{M}, \frac{w}{M}, \frac{w}{M}, \ldots, \frac{w}{M}\right)$ where w is given by

w = ($\pi'_1(\varphi_1 \otimes \phi_1)$, $\pi'_2(\varphi_2 \otimes \phi_2)$,……, $\pi'_{k*}(\varphi_{k*} \otimes \phi_{k*})$) (23)

Let $\varphi_i = (\varphi_{i,j})$ and $\phi_i = (\phi_{i,j})$ be the stationary probability components of the arrival and service processes. Neuts [5], gives the stability condition as, $w' A_0\, e < w' A_2\, e$ where w is given by (23). Taking the sum cross diagonally in the A_0 *and* A_2 matrices, it can be seen using (9) that w' $A_0\, e = \frac{1}{M}\, w(\sum_{n=1}^{M} n\Lambda_n)e = \frac{1}{M}\left(\sum_{n=1}^{M}\sum_{i=1}^{k*} n\,\pi'_i(\varphi_i \otimes \phi_i)(D_n^i \otimes I_{k'_i})e\right) = \frac{1}{M}\left(\sum_{n=1}^{M}\sum_{i=1}^{k*} n\,\pi'_i(\varphi_i D_n^i e \otimes \phi_i e)\right) = \frac{1}{M}\left(\sum_{i=1}^{k*}\pi'_i\,\varphi_i(\sum_{n=1}^{M} n\, D_n^i)e\right) < w' A_2\, e = \frac{1}{M}\, w(\sum_{n=1}^{N} nU_n)e = \frac{1}{M}\left(\sum_{n=1}^{N}\sum_{i=1}^{k*} n\,\pi'_i(\varphi_i \otimes \phi_i)(I_{k_i} \otimes S_n^i)e\right) = \frac{1}{M}\left(\sum_{n=1}^{N}\sum_{i=1}^{k*} n\,\pi'_i(\varphi_i e \otimes \phi_i S_n^i\, e)\right) = \frac{1}{M}\left(\sum_{i=1}^{k*}\pi'_i\phi_i(\sum_{n=1}^{N} n\, S_n^i)e\right)$. This gives the stability condition as $\sum_{i=1}^{k*}\pi'_i\ \varphi_i(\sum_{n=1}^{M} n\, D_n^i)e\ <\ \sum_{i=1}^{k*}\pi'_i\ \phi_i(\sum_{n=1}^{N} n\, S_n^i)e$ (24)

The sum $\varphi_i\ (\sum_{n=1}^{M} n\, D_n^i)e$ and $\phi_i(\sum_{n=1}^{N} n\, S_n^i)e$ are known as the fundamental rates or the stationary rates of arrival / service of the BMAP/ BMSP processes corresponding to the environment i for $1 \le i \le k*$. This result (24) is the stability condition for the random environment BMAP/BMSP/1 queue with random environment where maximum arrival size is greater than the maximum service size. When (24) is satisfied, the stationary distribution of the queue length exists Neuts [5]. Let π(0, i, j) : for $1 \le i \le k*$ $1 \le j \le k_i$); π(0, k, i, j, j') ; for $1 \le k \le M-1$; $1 \le i \le k*$; $1 \le j \le k_i$; $1 \le j \le k'_i$ and π(n, k, i, j, j'): for $0 \le k \le M-1$; $1 \le i \le k*$; $1 \le j \le k_i$; $1 \le j \le k'_i$ and $n \ge 1$ be the stationary probability vectors of Markov chain X(t) states in (1) for this model.

Let π_0=(π(0,1,1),π(0,1,2)…π(0,1,k_1),π(0,2,1),π(0,2,2)…π(0,2,k_2)…π(0,k*,1),π(0,k*,2)…π(0,k*,k_{k*}), π(0,1,1,1,1),π(0,1,1,1,2)…π(0,1,1,k_1, k'_1),π(0,1,2,1,1),π(0,1,2,1,2)…π(0,1,2,k_2, k'_2),π(0,1,3,1,1),π(0,1,3,1,2)… π(0,1,3,k_2, k'_2)…π(0,1,k*,1,1),π(0,1,k*,1,2)…π(0,1,k*,k_{k*}, k'_{k*}),π(0,2,1,1,1)…π(0,2,k*,k_{k*}, k'_{k*})… π(0,M-1,1,1,1),π(0,M-1,1,1,2)…π(0,M-1,k*,k_{k*}, k'_{k*})) be a vector of type 1x[$\sum_{i=1}^{k*} k_i + (M-1)\sum_{i=1}^{k*} k_i\, k'_i$].

Let π_n=(π(n,0,1,1,1),π(n,0,1,1,2)…π(n,0,1,k_1, k'_1),π(n,0,2,1,1),π(n,0,2,1,2)…π(n,0,2,k_2, k'_2),π(n,0,3,1,1), π(n,0,3,1,2)…π(n,0,3,k_3, k'_3)…π(n,0,k*,1,1),π(n,0,k*,1,2)…π(n,0,k*,k_{k*}, k'_{k*}),π(n,1,1,1,1),π(n,1,1,1,2)… π(n,1,1,k_1, k'_1),π(n,1,2,1,1),π(n,1,2,1,2)…π(n,1,2,k_2, k'_2),π(n,1,3,1,1),π(n,1,3,1,2)…π(n,1,3,k_3, k'_3)…π(n,1,k*,1 ,1),π(n,1,k*,1,2)…π(n,1,k*,k_{k*}, k'_{k*}),π(n,2,1,1,1)…π(n,2,k*,k_{k*}, k'_{k*}),π(n,3,1,1,1)…π(n,3,k*,k_{k*}, k'_{k*})… π(n,M-1,1,1,1),π(n,M-1,1,1,2)…π(n,M-1,k*,k_{k*}, k'_{k*})) be a vector of type 1x[$M\sum_{i=1}^{k*} k_i\, k'_i$]. The stationary probability vector $\pi = (\pi_0, \pi_1, \pi_3, \ldots)$ satisfies the equations πQ_A=0 and πe=1. (25)

From (25), it can be seen $\pi_0 B_1 + \pi_1 B_2$=0. (26)

$\pi_0 B_0 + \pi_1 A_1 + \pi_2 A_2 = 0$ (27)

$\pi_{n-1} A_0 + \pi_n A_1 + \pi_{n+1} A_2 = 0$, for $n \ge 2$. (28)

Introducing the rate matrix R as the minimal non-negative solution of the non-linear matrix equation $A_0 + RA_1 + R^2 A_2 = 0$, (29)

it can be proved (Neuts [5]) that π_n satisfies the following. $\pi_n = \pi_1 R^{n-1}$ for $n \ge 2$. (30)

Using (26), π_0 satisfies $\pi_0 = \pi_1 B_2(-B_1)^{-1}$ (31)

So using (27) and (31) and (30) the vector π_1 can be calculated up to multiplicative constant since π_1 satisfies the equation $\pi_1\,[B_2(-B_1)^{-1}B_0 + A_1 + RA_2]$ =0. (32)

Using (31) and (30) it can be seen that $\pi_1[B_2(-B_1)^{-1}e + (I-R)^{-1}e] = 1$. (33)

Replacing the first column of the matrix multiplier of π_1 in equation (32), by the column vector multiplier of π_1 in (33), a matrix which is invertible may be obtained. The first row of the inverse of that same matrix is π_1 and this gives along with (31) and (30) all the stationary probabilities of the system. The matrix R is iterated starting with $R(0) = 0$; and finding $R(n+1) = -A_0A_1^{-1} – R^2(n)A_2A_1^{-1}$, for n ≥ 0. The iteration may be terminated to get a solution of R at a norm level where $\|R(n+1) - R(n)\| < \varepsilon$.

2.3. Performance Measures of the System

(i) The probability of the queue length S = r > 0, P(S=r) can be seen as follows. For 1 ≤ r ≤ M-1, P(S =r) = $\sum_{i=1}^{k*}\sum_{j_1=1}^{k_i}\sum_{j_2=1}^{k'_1}\pi(0,r,i,j_1,j_2)$. For r ≥ M, let n and k be non negative integers such that r = n M + k. Then P(S=r) =$\sum_{i=1}^{k*}\sum_{j_1=1}^{k_i}\sum_{j_2=1}^{k'_i}\pi(n,k,i,j_1,j_2)$, where r = n M + k, n ≥ 1 and k ≥ 0. (34)

(ii) The probability that the queue length is zero is P(S =0) = $\sum_{i=1}^{k*}\sum_{j=1}^{k_i}\pi(0,i,j)$. (35)

(iii) The expected queue level E(S), can be calculated. Using (35) and (34), it may be seen that E(S)=$\sum_0^{\infty} r\,P(S=r)$=$\sum_{i=1}^{k*}\sum_{j=1}^{k_i}0\pi(0,i,j)$ +$\sum_{k=1}^{M-1}\sum_{i=1}^{k*}\sum_{j_1=1}^{k_i}\sum_{j_2=1}^{k'_i}k\pi(0,k,i,j_1,j_2)$
+$\sum_{n=1}^{\infty}\sum_{k=0}^{M-1}\sum_{i=1}^{k*}\sum_{j_1=1}^{k_i}\sum_{j_2=1}^{k'_i}\pi(n,k,i,j_1,j_2)(nM+k)$
=$\sum_{k=1}^{M-1}\sum_{i=1}^{k*}\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k'_i}k\pi(0,k,i,j_1,j_2)$ + $\sum_{n=1}^{\infty}\pi_n$.(Mn…Mn,Mn+1,…Mn+1,Mn+2,…Mn+2,…Mn+M-1,… Mn+M-1)=$\sum_{k=1}^{M-1}k\sum_{i=1}^{k*}\sum_{j_1=1}^{k_i}\sum_{j_2=1}^{k'_i}\pi(0,k,i,j_1,j_2)$ + M$\sum_{n=1}^{\infty}n\pi_n\,e$+$\pi_1(I-R)^{-1}\xi$.

Here ξ=$(0,\dots0,1,\dots,1,2,\dots,2,\dots,M-1,\dots,M-1)'$ is of type [($\sum_{i=1}^{k*}k_ik'_i$)M]x1 column vector in which consecutively ($\sum_{i=1}^{k*}k_ik'_i$) times 0,1,2,3..,M-1 appear. Let it be called ξ' when 0 appears ($\sum_{i=1}^{k*}k_i$)times and others in that order appear ($\sum_{i=1}^{k*}k_ik'_i$) times. Then E(S)= π_0ξ'+ $\pi_1(I-R)^{-1}\xi + M\pi_1(I-R)^{-2}e$ (36)

(iv)Variance of S can be derived. Let η be column vector η=$[0,\dots,0,1^2,\dots1^2\,2^2,\dots,2^2,\dots\,(M-1)^2,\dots,(M-1)^2]'$ of type [($\sum_{i=1}^{k*}k_ik'_i$) M]x1in which consecutively ($\sum_{i=1}^{k*}k_ik'_i$) times squares of 0,1,2,3.., M-1 appear. Let it be called η' when 0 appears ($\sum_{i=1}^{k*}k_i$) times and others in the same manner as in η appear ($\sum_{i=1}^{k*}k_ik'_i$) times. Then E(S^2)=$\sum_0^{\infty}r^2\,P(S=r)$=$\sum_{i=1}^{k*}\sum_{j=1}^{k_i}0\pi(0,i,j)$ +$\sum_{k=1}^{M-1}\sum_{i=1}^{k*}\sum_{j_1=1}^{k_i}\sum_{j_2=1}^{k'_i}\pi(0,k,i,j_1,j_2)k^2$
+$\sum_{n=1}^{\infty}\sum_{k=0}^{M-1}\sum_{i=1}^{k*}\sum_{j_1=1}^{k_i}\sum_{j_2=1}^{k'_i}\pi(n,k,i,j_1,j_2)(nM+k)^2$ =π_0η' + $M^2[\sum_{n=1}^{\infty}n(n-1)\pi_n\,e + \sum_{n=1}^{\infty}n\pi_n\,e]$ + $\sum_{n=1}^{\infty}\pi_n\eta$ + 2M$\sum_{n=1}^{\infty}n\,\pi_n\,\xi$.

So, E(S^2)= π_0η' + $M^2[\pi_1(I-R)^{-3}2R\,e + \pi_1(I-R)^{-2}e] + \pi_1(I-R)^{-1}\eta + 2M\pi_1(I-R)^{-2}\xi$ (37)

VAR(S)=E(S^2) $-[E(S)]^2$may be written from (36) and(37).

III. MODEL (B): MAXIMUM ARRIVAL SIZE M < MAXIMUM SERVICE SIZE N

The dual case of Model (A), namely the case, M < N is treated here. (When M =N both models are applicable and one can use any one of them.) The assumption (vii) of Model (A) is changed and all its other assumptions are retained.

3.1.Assumption

(vii) The maximum arrival size M= $\max_{1\le i\le k*}M_i$ is less than the maximum service size N=$\max_{1\le i\le k*}N_i$.

3.2.Analysis

Since this model is dual, the analysis is similar to that of Model (A). The differences are noted below. The state space of the chain is as follows presented in a similar way.

The state of the system of the continuous time Markov chain X (t) under consideration is presented as follows. X(t) = {(0, i, j) : for 1 ≤ i ≤ k* 1 ≤ j ≤ k_i)} U {(0, k, i, j, j') ; for 1 ≤ k ≤ N-1; 1 ≤ i ≤ k*; 1 ≤ j ≤ k_i; 1 ≤ j ≤ k'_i} U {(n, k, i, j, j'): for 0 ≤ k ≤ N-1; 1 ≤ i ≤ k*; 1 ≤ j ≤ k_i; 1 ≤ j ≤ k'_i and n ≥ 1}. (38)

The chain is in the state (0, i, j) when the number of customers in the queue is 0, the environment state is i for 1 ≤ i ≤ k*and the arrival phase is j for 1 ≤ j ≤ k_i. The chain is in the state (0, k, i, j, j') when the number of customers is k for 1 ≤ k ≤ N-1, the environment state is i for 1 ≤ i ≤ k*, the arrival phase is j for 1 ≤ j ≤ k_i and the service phase is j' for 1 ≤ j' ≤ k'_i. The chain is in the state (n, k, i, j, j') when the number of customers in the queue is n N + k, for 0 ≤ k ≤ N-1 and 1 ≤ n < ∞, the environment state is i for 1 ≤ i ≤ k*, the arrival phase is j for 1 ≤ j ≤ k_i and the service phase is j' for 1 ≤ j' ≤ k'_i. When the number of customers waiting in the system is r, then r is identified with (n, k) where r on division by N gives n as the quotient and k as the remainder. The infinitesimal generator Q_B of the model has the same block partitioned structure given in (4) for Model (A) but the inner matrices are of different orders.

$$Q_B = \begin{bmatrix} B'_1 & B'_0 & 0 & 0 & . & . & . & \cdots \\ B'_2 & A'_1 & A'_0 & 0 & . & . & . & \cdots \\ 0 & A'_2 & A'_1 & A'_0 & 0 & . & . & \cdots \\ 0 & 0 & A'_2 & A'_1 & A'_0 & 0 & . & \cdots \\ 0 & 0 & 0 & A'_2 & A'_1 & A'_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{39}$$

In (39) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}$, ... For partition purpose the zero states in the first two sets of equation (38) are combined. The vector $\underline{0}$ is of type 1x$[\sum_{i=1}^{k*} k_i + (N-1)\sum_{i=1}^{k*} k_i\, k'_i]$with
$\underline{0}$=((0,1,1),(0,1,2),(0,1,3)…(0,$1, k_1$),(0,2,1),(0,2,2),(0,2,3)…(0,$2, k_2$)…(0,k*,1),(0,k*,2),(0,k*,3)…(0,k*,k_{k*}), (0,1,1,1,1),(0,1,1,1,2)…(0,1,1,1, k'_1),(0,1,1,2,1),(0,1,1,2,2)...(0,1,1,2,k'_1),(0,1,1,3,1)...(0,1,1,3,k'_1)…(0,1,1,k_1,1) …(0,1,1,k_1, k'_1),(0,1,2,1,1),(0,1,2,1,2)…(0,1,2,1, k'_2),(0,1,2,2,1),(0,1,2,2,2)….(0,1,2,2,k'_2),(0,1,2,3,1)....(0,1,2,3, k'_2)…(0,1,2,k_2,1)…(0,1,2,k_2, k'_2),(0,1,3,1,1)…(0,1,3,k_3, k'_3)…(0,1,k*,1,1),…,(0,1,k*,k_{k*}, k'_{k*}),(0,2,1,1,1),(0,2, 1,1,2)…(0,2,k*,k_{k*},k'_{k*}),(0,3,1,1,1)…(0,3,k*,$k_{k*}k_{k*}$),(0,4,1,1,1)…(0,4,k*,$k_{k*}k'_{k*}$)…(0,N-1,1,1,1)… (0,N-1,k*,k_{k*}, k'_{k*})) and the vector $\underline{n}$ is of type 1x$[N\sum_{i=1}^{k*} k_i\, k'_i]$ and is given in a similar manner as follows $\underline{n}$=(n,0,1,1,1),(n,0,1,1,2)...(n,0,1,1, k'_1),(n,0,1,2,1)…(n,0,1,2,k'_1)…(n,0,k*,1,1)…(n,0,k*,k_{k*}, k'_{k*}),(n,1,1,1,1)… (n,1,k*,k_{k*},k'_{k*}),(n,2,1,1,1)...(n,2,k*,k_{k*},k'_{k*})…(n,N-1,1,1,1),(n,N-1,1,1,2)…(n,N-1,k*,k_{k*}, k'_{k*})).

The matrices$B'_1 and\ A'_1$ have negative diagonal elements, they are of orders $[\sum_{i=1}^{k*} k_i + (N-1)\sum_{i=1}^{k*} k_i\, k'_i]$ and $[N\sum_{i=1}^{k*} k_i\, k'_i]$ respectively and their off diagonal elements are non-negative. The matrices $A'_0\, and A'_2$ have nonnegative elements and are of order $[\, N\sum_{i=1}^{k*} k_i\, k'_i\,]$. The matrices $B'_0\ and\ B'_2$ have non-negative elements and are of types $[\sum_{i=1}^{k*} k_i + (N-1)\sum_{i=1}^{k*} k_i\, k'_i]$ x $[\ N\sum_{i=1}^{k*} k_i\, k'_i\ ]$ and $[\ N\sum_{i=1}^{k*} k_i\, k'_i\ ]$ x $[\sum_{i=1}^{k*} k_i + (N-1)\sum_{i=1}^{k*} k_i\, k'_i]$ and they are given below. Using Model (A) for definitions of $\Lambda_j\ , \Lambda'_j,\ U_j, V_j$ and U and letting Ω and Ω' as in Model (A), the partitioning matrices are defined as follows. The matrix $B'_0$is same as that of A'_0 with first zero block row is of order $[\,\sum_{i=1}^{k*} k_i]$ x$[\ N\sum_{i=1}^{k*} k_i\, k'_i]$. The matrix $B'_2$is same as that of A'_2 except the first column block is of type $[\ N\sum_{i=1}^{k*} k_i\, k'_i\ ]$ x $[\,\sum_{i=1}^{k*} k_i]$ and is $(U'_N, 0, \ldots .0)'$ where

$$U'_N = \begin{bmatrix} I_{k_1} \otimes S_N^1 e & 0 & 0 & \cdots & 0 \\ 0 & I_{k_2} \otimes S_N^2 e & 0 & \cdots & 0 \\ 0 & 0 & I_{k_3} \otimes S_N^3 e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{k_{k*}} \otimes S_N^{k*} e \end{bmatrix} \tag{40}$$

$$A'_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \Lambda_M & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 \end{bmatrix} \tag{41}$$

$$A'_2 = \begin{bmatrix} U_N & U_{N-1} & U_{N-2} & \cdots & U_3 & U_2 & U_1 \\ 0 & U_N & U_{N-1} & \cdots & U_4 & U_3 & U_2 \\ 0 & 0 & U_N & \cdots & U_5 & U_4 & U_3 \\ 0 & 0 & 0 & \ddots & U_6 & U_5 & U_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_N & U_{N-1} & U_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & U_N & U_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & U_N \end{bmatrix} \tag{42}$$

$$A'_1 = \begin{bmatrix} \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 & 0 \\ U_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 & 0 \\ U_2 & U_1 & \Omega & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_{N-M-1} & U_{N-M-2} & U_{N-M-3} & \cdots & \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M \\ U_{N-M} & U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ U_{N-M+1} & U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 & \Omega & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_{N-2} & U_{N-3} & U_{N-4} & \cdots & U_{N-M-2} & U_{N-M-3} & U_{N-M-2} & \cdots & \Omega & \Lambda_1 \\ U_{N-1} & U_{N-2} & U_{N-3} & \cdots & U_{N-M-1} & U_{N-M-2} & U_{N-M-1} & \cdots & U_1 & \Omega \end{bmatrix} \quad (43)$$

$$B'_1 = \begin{bmatrix} \Omega' & \Lambda'_1 & \Lambda'_2 & \cdots & \Lambda'_M & 0 & 0 & \cdots & 0 & 0 \\ U & \Omega & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 & 0 \\ V_1 & U_1 & \Omega & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ V_{N-M-2} & U_{N-M-2} & U_{N-M-3} & \cdots & \Omega & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M \\ V_{N-M-1} & U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & \Omega & \Lambda_1 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ V_{N-M} & U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 & \Omega & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ V_{N-3} & U_{N-3} & U_{N-4} & \cdots & U_{N-M-2} & U_{N-M-3} & U_{N-M-2} & \cdots & \Omega & \Lambda_1 \\ V_{N-2} & U_{N-2} & U_{N-3} & \cdots & U_{N-M-1} & U_{N-M-2} & U_{N-M-1} & \cdots & U_1 & \Omega \end{bmatrix} \quad (44)$$

$$Q''_B = \begin{bmatrix} \Omega + U_N & \Lambda_1 + U_{N-1} & \cdots & \Lambda_{M-1} + U_{N-M+1} & \Lambda_M + U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 \\ U_1 & \Omega + U_N & \cdots & \Lambda_{M-2} + U_{N-M+2} & \Lambda_{M-1} + U_{N-M+1} & \Lambda_M + U_{N-M} & \cdots & U_3 & U_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ U_{N-M-2} & U_{N-M-3} & \cdots & \Omega + U_N & \Lambda_1 + U_{N-1} & \Lambda_2 + U_{N-2} & \cdots & \Lambda_M + U_{N-M} & U_{N-M-1} \\ U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & \Omega + U_N & \Lambda_1 + U_{N-1} & \cdots & \Lambda_{M-1} + U_{N-M+1} & \Lambda_M + U_{N-M} \\ \Lambda_M + U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 & \Omega + U_N & \cdots & \Lambda_{M-2} + U_{N-M+2} & \Lambda_{M-1} + U_{N-M+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_2 + U_{N-2} & \Lambda_3 + U_{N-3} & \cdots & U_{N-M-1} & U_{N-M-2} & U_{N-M-3} & \cdots & \Omega + U_N & \Lambda_1 + U_{N-1} \\ \Lambda_1 + U_{N-1} & \Lambda_2 + U_{N-2} & \cdots & \Lambda_M + U_{N-M} & U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & \Omega + U_N \end{bmatrix} \quad (45)$$

The basic generator which is concerned with only the arrival and service is $Q''_B = A'_0 + A'_1 + A'_2$. This is also block circulant. Using similar arguments given for Model (A) it can be seen that its probability vector is w' = $\left(\frac{w}{N}, \frac{w}{N}, \frac{w}{N}, \ldots\ldots, \frac{w}{N}\right)$ where w is given by (23) and the stability condition remains the same. Following the arguments given for Model (A), one can find the stationary probability vector for Model (B) also in matrix geometric form. All performance measures including expectation of customers waiting for service and its variance for Model (B) have the form as in Model (A) except M is replaced by N.

IV. NUMERICAL ILLUSTRATION

Numerical examples are presented for five cases, namely, (i) M=N=4; (ii) M=4,N=3 ; (iii) M=4, N=2; (iv) M=3, N=4and (v) M=2, N=4. There are two environments with generator $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. For the case (i) M=N=4 the batch Markovian arrival and service processes (BMAP and BMSP) have representations given by $\{D_n^i : 0 \le n \le 4$ and i=1, 2 } and $\{S_n^i : 0 \le n \le 4$ and i=1, 2}respectively where the BMAP arrival matrices are for environment 1, $D_0^1 = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -4 & 2 \\ 3 & 1 & -5 \end{bmatrix}$, $D_1^1 = \begin{bmatrix} .15 & .15 & .2 \\ .18 & .18 & .24 \\ .21 & .21 & .28 \end{bmatrix}$, $D_2^1 = \begin{bmatrix} .09 & .09 & .12 \\ .09 & .09 & .12 \\ .06 & .06 & .08 \end{bmatrix}$, $D_3^1 = \begin{bmatrix} .03 & .03 & .04 \\ .03 & .03 & .04 \\ .03 & .03 & .04 \end{bmatrix}$, $D_4^1 = \begin{bmatrix} .03 & .03 & .04 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and for environment 2, $D_0^2 = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -4 & 2 \\ 3 & 0 & -5 \end{bmatrix}$, $D_1^2 = \begin{bmatrix} .48 & .48 & .24 \\ .4 & .4 & .2 \\ .4 & .4 & .2 \end{bmatrix}$, $D_2^2 = \begin{bmatrix} .24 & .24 & .12 \\ .32 & .32 & .16 \\ .24 & .24 & .12 \end{bmatrix}$, $D_3^2 = \begin{bmatrix} .08 & .08 & .04 \\ .08 & .08 & .04 \\ .16 & .16 & .08 \end{bmatrix}$, and $D_4^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The service BMSP matrices are for environment 1, $S_0^1 = \begin{bmatrix} -4 & 1 \\ 1 & -5 \end{bmatrix}$, $S_1^1 = \begin{bmatrix} .6 & .9 \\ .96 & 1.44 \end{bmatrix}$, $S_2^1 = \begin{bmatrix} .36 & .54 \\ .48 & .72 \end{bmatrix}$, $S_3^1 = \begin{bmatrix} .12 & .18 \\ .16 & .24 \end{bmatrix}$, and $S_4^1 = \begin{bmatrix} .12 & .18 \\ 0 & 0 \end{bmatrix}$ and for environment 2, $S_0^2 = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix}$, $S_1^2 = \begin{bmatrix} 1.08 & .72 \\ .3 & .2 \end{bmatrix}$, $S_3^2 = \begin{bmatrix} .18 & .12 \\ .06 & .24 \end{bmatrix}$ and $S_4^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ The starting probability vectors of service processes for the two environments on arrival of customers when the queue is empty are β_1=(.4, .6) and β_2=(.6, .4). For the case (ii)

M=4, N=3 the above batch arrival and batch service rates (matrices) of case (i) are assumed except two service rates matrices which are replaced as $S_3^1=\begin{bmatrix} .24 & .36 \\ .16 & .24 \end{bmatrix}$, and $S_4^1=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For the case (iii) M=4, N=2 the above batch arrival and batch service rates (matrices) of case (i) are assumed except the six service rates matrices which are assumed as $S_2^1=\begin{bmatrix} .6 & .9 \\ .64 & .96 \end{bmatrix}$, $S_2^2=\begin{bmatrix} .72 & .48 \\ .3 & .2 \end{bmatrix}$, $S_3^1=S_4^1$=0 matrices and $S_3^2=S_4^2$=0 matrices. For the case (iv) M=3, N=4 the above batch arrival and batch service rates matrices of case (i) are assumed except two arrival rates matrices which are replaced as $D_3^1=\begin{bmatrix} .06 & .06 & .08 \\ .03 & .03 & .04 \\ .03 & .03 & .04 \end{bmatrix}$and $D_4^1=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. For the case (v) M=2, N=4 the above batch arrival and batch service rates (matrices) of case (i) are assumed except the six arrival rates matrices which are assumed as $D_2^1=\begin{bmatrix} .15 & .15 & .2 \\ .12 & .12 & .16 \\ .09 & .09 & .12 \end{bmatrix}$, $D_2^2=\begin{bmatrix} .32 & .32 & .16 \\ .4 & .4 & .2 \\ .4 & .4 & .2 \end{bmatrix}$, $D_3^1=D_4^1$=0 matrices and $D_3^2=D_4^2$=0 matrices. The partitioned matrices are of order 48, the rate matrix R is of order 48 and fifteen iterations are performed to evaluate R matrix. The results obtained for various performance measures are tabulated in table 1. Queue length probabilities and the probabilities of batch sizes are estimated. These probabilities and expected queue lengths show variations depending on arrival, service rates and batches of arrival and service sizes. The figures (1) and (2) present the probabilities of various levels and blocks.

Table 1. Results Obtained for the Five Cases.

	M=4=N	M=4,N=3	M=4,N=2	M=3,N=4	M=2,N=4
P(S=0)	0.380758062	0.377246667	0.348733946	0.3849806	0.421313996
P(S=1)	0.129820258	0.129530535	0.125595408	0.131199748	0.143065373
P(S=2)	0.110187529	0.110124761	0.108434643	0.111159883	0.131171064
P(S=3)	0.086386188	0.086540194	0.086908008	0.088387612	0.081273082
π0e	0.707152037	0.703442157	0.669672005	0.715727843	0.776823515
π1e	0.185713263	0.187070777	0.197490543	0.181491843	0.159704069
π2e	0.067545585	0.068669501	0.078987097	0.06535554	0.045203898
π3e	0.024943425	0.025583927	0.032001528	0.023784268	0.013006974
π(n ≥ 4)e	0.01464569	0.015233637	0.021848827	0.013640506	0.005261545
Arr rate	0.59475	0.59475	0.59475	0.588958333	0.540481481
Ser rate	1.110243056	1.0921875	1.004108796	1.110243056	1.110243056
Norm	4.82513E-05	5.40723E-05	0.000113132	4.49819E-05	1.49854E-05
E(S)	2.621582695	2.656880644	2.992491652	2.559407301	2.030074169
Std(S)	3.863770414	3.904750478	4.318346577	3.787733035	3.021600968

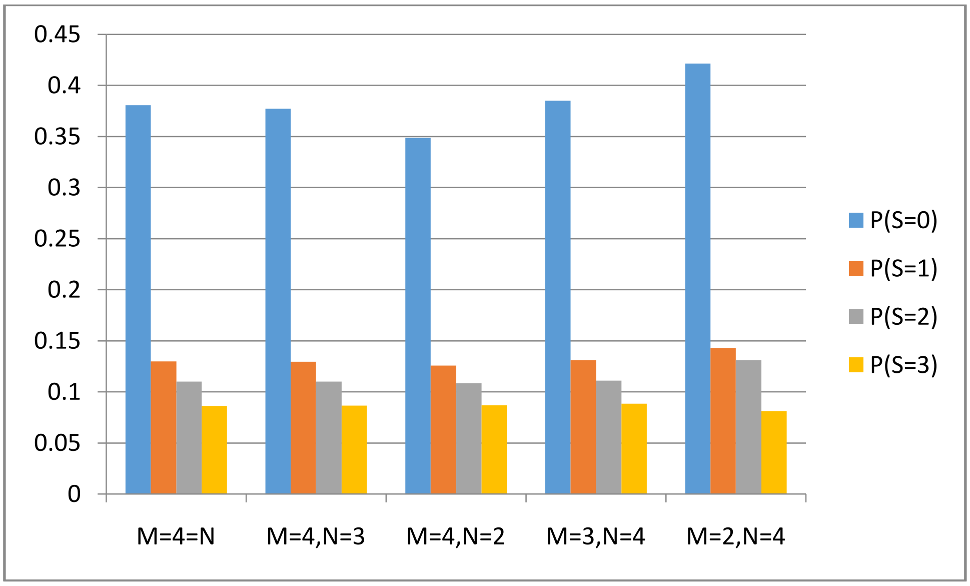

Figure 1. Probabilities of Queue Lengths

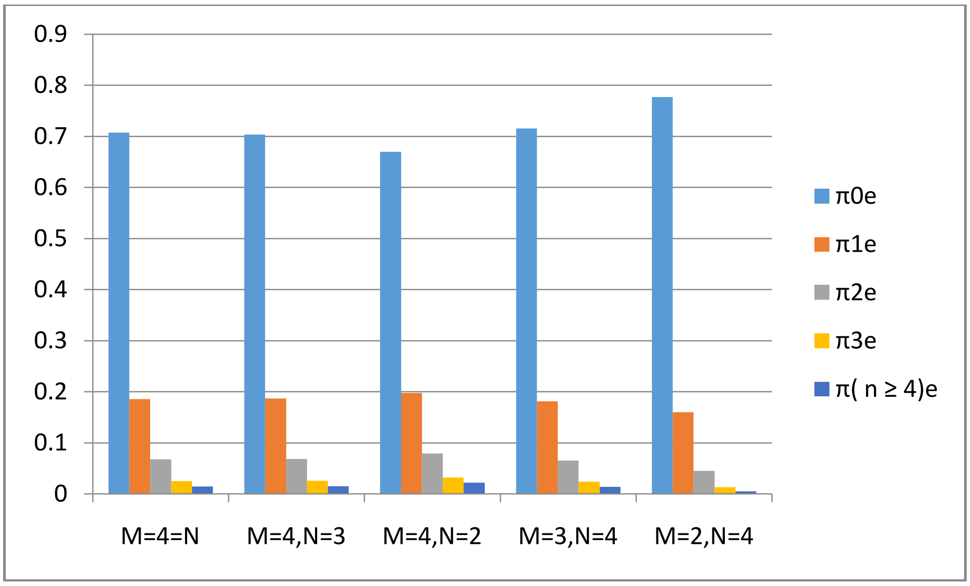

Figure.2. The probabilities of customer blocks

V. CONCLUSION

Two BMAP/BMSP/1 bulk arrival and bulk service queues with random environment have been studied by identifying the maximum of the arrival and the service sizes and by grouping the customers as members of blocks of such maximum sizes. Matrix geometric results have been obtained by partitioning the infinitesimal generator by grouping of customers, environment state BMAP and BMSP phases together. The basic system generators of the queues are block circulant matrices which are explicitly presenting the stability condition in standard forms. Numerical results for bulk queue models are presented and discussed. Effects of variation of rates on expected queue length and on probabilities of queue lengths are exhibited. The decrease in arrival rates (so also increase in service rates) makes the convergence of R matrix faster which can be seen in the decrease of norm values. The standard deviations also decrease. The BMAP/BMSP/1 queue with random environment has number of applications. The process includes Exponential, Erlang, Hyper Exponential, Coxian distributions and PH distributions as special cases and the PH distribution is also a best approximation for a general distribution. Further the BMAP/BMSP/1 queue is a most general form almost equivalent to G/G/1 queue. The bulk arrival models because they have non-zero elements or blocks above the super diagonals in infinitesimal generators, they require for studies the decomposition methods with which queue length probabilities of the system are written in a recursive manner. Their applications are much limited compared to matrix geometric results. From the results obtained here, provided the maximum arrival and service sizes are not infinity, the most general model of the BMAP/BMSP/1 queue with random environment admits matrix geometric solution. Further studies with block circulant basic generator system may produce interesting and useful results in inventory theory and finite storage models like dam theory. It is also noticed here that once the maximum arrival or service size increases, the order of the rate matrix increases proportionally. However the matrix geometric structure is retained and rates of convergence is not much affected. Inventory models with BMAP and BMSP assumptions may be focused for further study which may produce more general results.

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REFERENCES

[1] Bini.D,Latouche.G,and-Meini.B, Numerical methods for structured Markov chains, Oxford Univ. Press, 2005.
[2] Chakravarthy.S.R and Neuts. M.F, Analysis of a multi-server queue model with MAP arrivals of special customers,SMPT,-Vol.43,2014,pp.79-95,
[3] Gaver, D., Jacobs, P., Latouche, G, Finite birth-and-death models in randomly changing environments. AAP.16,1984,pp.715–731
[4] Latouche.G, and Ramaswami.V, Introduction to Matrix Analytic Methods in Stochastic Modeling, SIAM. Philadelphia.1998

[5] Neuts.M.F,Matrix-Geometric Solutions in Stochastic Models: An algorithmic Approach, The Johns Hopkins Press, Baltimore,1981.
[6] Rama Ganesan, Ramshankar.R, and Ramanarayanan.R, M/M/1 Bulk Arrival and Bulk Service Queue with Randomly Varying Environment,IOSR-JM,Vol.10,Issue6,Ver.III,2014,pp58-66.
[7] Sandhya.R, Sundar.V, Rama.G, Ramshankar.R and Ramanarayanan.R, M/M//C Bulk Arrival And Bulk Service Queue With Randomly Varying Environment,IOSR-JEN,Vol.05,Issue02,||V1||,2015,pp.13-26.
[8] Ramshankar.R, Rama Ganesan, and Ramanarayanan.R PH/PH/1 Bulk Arrival and Bulk Service Queue, IJCA,Vol.109,No.3,2015,pp.27-33.
[9] Ramshankar.R, Rama.G, Sandhya.R, Sundar.V and Ramanarayanan.R,(2015),PH/PH/1 Bulk Arrival and Bulk Service Queue with Randomly Varying-Environment,-IOSRJEN,Vol.05.Issue02-||V4||.pp.01-12.
[10] Rama.G, Ramshankar.R, Sandhya.R, Sundar.V and Ramanarayanan.R, (2015),BMAP/M/C Bulk Service Queue with Randomly Varying Environment, IOSRJEN, Vol.05, Issue 03,||V1||,pp.33-47.
[11] Lucantoni.D.M, (1993), Models and Techniques for Performance Evaluation of Computer and Communication Systems,L. Donatiello and R. Nelson-Editors-Springer-verlag,,-pp330-358.
[12] Cordeiro and Kharoufch, J.P, Batch Markovian Arrival Processes (BMAP),www.dtic.mil/get-tr-doc/pdf/AD=ADA536697&origin=publication-Detail.
[13] Qi-Ming-He,Fundamentals-of-Matrix-Analytic-Methods,Springer,2014.
[14] Rama Ganesan, and Ramanarayanan R, 2014, Stochastic Analysis of Project Issues and Fixing its Cause by Project Team and Funding System Using-Matrix-Analytic-Method,IJCA,VOL.107,No.7,2014,pp.22-28.
[15] Usha.K, Contribution to The Study of Stochastic Models in Reliability and Queues, Ph.D Thesis, 1981, Annamlai University,India.
[16] Usha.K, The PH/M/C Queue with Varying Environment,Zastosowania Matematyki Applications Mathematicae,XVIII,2 1984, pp.169-175.